Algebraic Proofs GCSE Higher Tier A/A*Grades KS4 with Answers/Solutions

1. Prove that $(n+4)^2-(3n+4)=(n+1)(n+4)+8$
2. Prove that $(n+4)^2-(3n+4)=(n+2)(n+3)+6$
3. Prove that $(n+3)^2-(3n+5)=(n+1)(n+2)+2$
4. Prove that $(n-5)^2-(2n-1)=(n-3)(n-9)-1$
5. Prove that $(n-3)^2-(n-5)=(n-3)(n-4)+2$
6. Prove that $\frac{1}{2}(n+1)(n+2)-\frac{1}{2}n(n+1)=n+1$
7. Prove that $\frac{1}{4}(2n+1)(n+4)-\frac{1}{4}n(2n+1)=2n+1$

8. Prove that $(3n+1)^2-(3n-1)^2$ is a multiple of 6 for all positive integer values of n.
9. Prove that $(4n+1)^2-(4n-1)^2$ is a multiple of 8 for all positive integer values of n.
10. Prove that $(5n+1)^2-(5n-1)^2$ is a multiple of 5 for all positive integer values of n.

11. Prove that $(2n+1)^2-(2n-1)^2$ is a multiple of 8 for all positive integer values of n.
12. Prove that $(5n+1)^2-(5n-1)^2$ is a multiple of 4 for all positive integer values of n.
13. Prove that $(2n+1)^2-(2n-1)^2-10$ is not a multiple of 8 for all positive integer values of n.
14. Prove that $(2n+1)^2-(2n-1)^2-2$ is not a multiple of 4 for all positive integer values of n.
15. Prove that $(n+1)^2-(n-1)^2+1$ is always odd for all positive integer values of n.
16. Prove that $(n+1)^2-(n-1)^2+4$ is always even for all positive integer values of n.

In all the questions below, n is a positive integer.

17. If $2n$ is always even for all positive integer values of n,
prove algebraically that the sum of the squares of any two consecutive even numbers is always a multiple of 4.

18. If $(2n + 1)$ is always odd for all positive integer values of n,
prove algebraically that the sum of the squares of any two consecutive odd numbers cannot be a multiple of 4.

19. Prove algebraically that the sum of the squares of any two consecutive numbers always leaves a remainder of 1 when divided by 4.

20. Prove algebraically that the difference between the squares of any two consecutive even numbers is always a multiple of 4.

21. Prove algebraically that the difference between the squares of any two consecutive odd numbers is always a multiple of 8.

22. Prove algebraically that the difference between the squares of any two consecutive numbers is always an odd number

23. Prove algebraically that the sum of the squares of any three consecutive even numbers always a multiple of 4.

24. Prove algebraically that the sum of the squares of any three consecutive odd numbers always leaves a remainder of 11 when divided by 12.

Answers/Solutions:

① $(n+4)^2-(3n+4)$

$= n^2+8n+16-3n-4$

$= n^2+5n+12$

$= n^2+5n+4+8$

$= (n+1)(n+4)+8 = R.H.S$

② $(n+4)^2-(3n+4)$

$= n^2+8n+16-3n-4$

$= n^2+5n+12$

$= n^2+5n+6+6$

$= (n+2)(n+3)+6$

$= RHS$

③ $(n+3)^2-(3n+5)$

$= n^2+6n+9-3n-5$

$= n^2+3n+4$

$= n^2+3n+2+2$

$= (n+1)(n+2)+2 = R.H.S$

④ $(n-5)^2-(2n-1)$

$= n^2-10n+25-2n+1$

$= n^2-12n+26$

$= n^2-12n+27-1$

$= (n-3)(n-9)-1 = RHS$

⑤ $(n-3)^2-(n-5)$

$= n^2-6n+9-n+5$

$= n^2-7n+14$

$= n^2-7n+12+2$

$= (n-3)(n-4)+2 = RHS$

⑥ $\frac{1}{2}(n+1)(n+2)-\frac{1}{2}n(n+1)$

$= \frac{1}{2}(n+1)[n+2-n]$

$= \cancel{\frac{1}{2}}(n+1)(\cancel{2})$

$= n+1 = R.H.S$

⑦ $\frac{1}{4}(2n+1)(n+4)-\frac{1}{4}n(2n+1)$

$=\frac{1}{4}(2n+1)(n+4-n)$

$=\frac{1}{\cancel{4}_1}(2n+1)(\cancel{4}^1) = 2n+1 = R.H.S$

⑧ $(3n+1)^2-(3n-1)^2$

$= 9n^2+6n+1-(9n^2-6n+1)$

$= 9n^2+6n+1-9n^2+6n-1$

$= 12n = 6\times 2n$ and always divisible by 6, hence a multiple of 6.

OR use the difference of 2 squares $p^2-q^2=(p+q)(p-q)$

$[3n+1+3n-1][3n+1-(3n-1)]$

$=[6n][3n+1-3n+1]$

$=[6n][2] = 12n = 6\times 2n$ etc.

⑨ $(4n+1)^2-(4n-1)^2$

$= 16n^2+8n+1-(16n^2-8n+1)$

$= 16n^2+8n+1-16n^2+8n-1$

$= 16n = 8\times 2n$ always divisible hence a multiple of 8.

OR $[4n+1+4n-1][4n+1-(4n-1)]$

$= [8n][4n+1-4n+1]$

$= [8n][2]$

$= 8\times 2n$ etc.

⑩ $(5n+1)^2-(5n-1)^2$

$= 25n^2+10n+1-(25n^2-10n+1)$

$= 25n^2+10n+1-25n^2+10n-1$

$= 20n = 5\times 4m$, always divisible by 5 and hence a multiple of 5.

OR $[5n+1+5n-1][5n+1-(5n-1)]$

$=[10n][5n+1-5n+1]$

$=[10n][2] = 20n$ etc.

(11) $(2n+1)^2 - (2n-1)^2$

$= 4n^2 + 4n + 1 - (4n^2 - 4n + 1)$

$= 4n^2 + 4n + 1 - 4n^2 + 4n - 1$

$= 8n$ which is a multiple of 8.

OR $[2n+1+2n-1][2n+1-(2n-1)]$

$= [4n][2n+1-2n+1]$

$= [4n][2] = 8n$ etc.

(12) Refere to Q10 which resulted in $20n = 4 \times 5n$ and hence a multiple of 4.

(13) From (11)

$(2n+1)^2 - (2n-1)^2 = 8n$

hence $(2n+1)^2 - (2n-1)^2 - 10$

$= 8n - 10 = 8n - 8 - 2$

$= 8(n-1) - 2$

$8(n-1)$ is divisible by 8 and hence a multiple of 8, but -2 is not a multiple of 8.

(14) From (11)

$(2n+1)^2 - (2n-1)^2 = 8n$

hence $(2n+1)^2 - (2n-1)^2 - 2$

$= 8n - 2$ $8n$ is a multiple of 4 but -2 is not.

(15) $(n+1)^2 - (n-1)^2 + 1$

$= n^2 + 2n + 1 - (n^2 - 2n + 1) + 1$

$= n^2 + 2n + 1 - n^2 + 2n - 1 + 1$

$= 4n + 1$ $4n$ is always divisible by 2 and hence even

even + 1 = odd.

(16) from (15) $(n+1)^2 - (n-1)^2 + 4$

$= 4n + 4 = 2(2n+2)$

divisible by 2 and hence always even.

(17) $(2n)^2 + (2n+2)^2$

$= 4n^2 + 4n^2 + 8n + 4$

$= 8n^2 + 8n + 4$

$= 4(2n^2 + 2n + 1)$ divisible by 4 and hence always a multiple of 4.

(18) $(2n+1)^2 + (2n+3)^2$

$= 4n^2 + 4n + 1 + 4n^2 + 12n + 9$

$= 8n^2 + 16n + 10$

$= 8n^2 + 16n + 8 + 2$

$= \underbrace{4(2n^2 + 8n + 2)}_{\text{divisible by 4 and a multiple of 4}} + 2 \rightarrow$ is not a multiple of 4

OR use $(2n-1)^2 + (2n+1)^2$

$= 4n^2 - 4n + 1 + 4n^2 + 4n + 1$

$= \underbrace{8n^2}_{\text{multiple of 4}} + 2 \rightarrow$ not a multiple of 4.

(19) $(2n)^2 + (2n+1)^2$

$= 4n^2 + 4n^2 + 4n + 1$

$= 8n^2 + 4n + 1$

$= \underbrace{4(2n^2 + n)}_{\text{a multiple of 4 and divisible by 4}} + 1 \leftarrow$ Remainder

$\dfrac{4(2n^2+n)+1}{4}$

$= \dfrac{\cancel{4}(2n^2+n)}{\cancel{4}} + \dfrac{1}{4}$

Remainder is 1

OR $\dfrac{8n^2}{4} + \dfrac{4n}{4} + \dfrac{1}{4}$

$= 2n^2 + n + \dfrac{1}{4}$

(20) $(2n+2)^2 - (2n)^2$

$= 4n^2 + 8n + 4 - 4n^2$

$= 8n + 4 = 4(2n+1)$ is a multiple of 4.

(21) $(2n+3)^2 - (2n+1)^2$

$= 4n^2 + 12n + 9 - (4n^2 + 4n + 1)$

$= 4n^2 + 12n + 9 - 4n^2 - 4n - 1$

$= 8n + 8$

$= 8(n+1)$ a multiple of 8.

OR use $(2n+1)^2 - (2n-1)^2$

$= 4n^2 + 4n + 1 - (4n^2 - 4n + 1)$

$= 4n^2 + 4n + 1 - 4n^2 + 4n - 1$

$= 8n$ a multiple of 8.

Note you may use the difference of 2 squares and factorise to get the answer.

(22) $(2n+1)^2 - (2n)^2$

$= 4n^2+4n+1-4n^2 = 4n+1$ (See Q19) 4n is even, 4n+1 is odd.

OR

$(n+1)^2 - n^2 = n^2+2n+1-n^2 = 2n+1$ odd.

(23) $(2n)^2 + (2n+2)^2 + (2n+4)^2$

$= 4n^2 + 4n^2 + 8n + 4 + 4n^2 + 16n + 16$

$= 12n^2 + 24n + 20$

$= 4(3n^2+6n+5)$

a multiple of 4.

(24) $(2n+1)^2 + (2n+3)^2 + (2n+5)^2$

$= 4n^2 + 4n + 1$
$+ 4n^2 + 12n + 9$
$+ 4n^2 + 20n + 25$

$= 12n^2 + 36n + 35$

$= 12n^2 + 36n + 24 + 11$

$= \underbrace{12(n^2+3n+2)}_{\text{divisible by } 12} + 11 \leftarrow$ Remainder.

Note If n is a positive integer then $2n$ is always even and $2n+1$ is always odd.

I hope you find this useful and that it challenges some of your students. My solutions are not unique.

Some other solutions may require re-wording some questions (19-24).

Apologies for the presentation! If you find any errors, please let me know.