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A20 FIND APPROXIMATE SOLUTIONS TO EQUATIONS NUMERICALLY USING ITERATION (higher tier)

ITERATION

Iteration is the act of repeating a process, either to generate an unbounded sequence of outcomes, or with the aim of approaching a desired goal, target or result.

Each repetition of the process is also called an iteration, and the results of one iteration are used as the starting point for the next iteration.

Iteration is a way of solving equations. It is often used as a means of obtaining successively closer approximations to the solution of a problem. You would usually use iteration when you cannot solve the equation any other way.

NOTATION

x_n is the *n*th term of a sequence.

x_{n+1} is the $(n + 1)$th term i.e. the term after the *n*th term.

x_{n+2} is the $(n + 2)$th term i.e. the second term after the *n*th term. This is also the term after the $(n + 1)$th term.

x_{n-1} is the $(n - 1)$th term i.e. the term before the *n*th term.

x_0 is often used as the starting point of an iterative solution

x_1 is the first term of a sequence

TYPES OF SEQUENCES

Sequences may be defined by having enough terms to establish a pattern, for example, 1, 2, 4, 8, 16

Sequences may be defined by having a rule, for example, *n*th term = $4n + 3$ producing 7, 11, 15, 19, 23

Sequences may be defined by giving the first term and the relationship between consecutive terms. The next example shows how this type of sequences works.

An **iterative sequence** is defined by giving the relationship between consecutive terms, e.g. The relationship, $x_{n+1} = x_n - 3$, is said to produce an iterative sequence.

EXAMPLE 1

Find the next five terms of the sequence given by $x_{n+1} = x_n - 3$ where $x_1 = 10$

$x_{n+1} = x_n - 3$ so we can say that:
any term = the term before it − 3

It is useful to put the equation in words where x_{n+1} = any term and x_n is the term before it

$x_2 = x_1 - 3 = 10 - 3 = 7$ — First term = 10; 2nd term = 1st term − 3

$x_3 = x_2 - 3 = 7 - 3 = 4$ — 3rd term = 2nd term − 3

$x_4 = x_3 - 3 = 4 - 3 = 1$ — 4th term = 3rd term − 3

$x_5 = x_4 - 3 = 1 - 3 = -2$ — 5th term = 4th term − 3

$x_6 = x_5 - 3 = -2 - 3 = -5$ — 6th term = 5th term − 3

Next five terms = 7, 4, 1, −2, −5

EXERCISE 1:

1. Write down the next five terms of the following sequences.

(a) $x_{n+1} = x_n + 2$ and $x_1 = 3$

(b) $x_{n+1} = 3x_n - 1$ and $x_1 = 2$

(c) $x_{n+1} = 2x_n - 1$ and $x_1 = \frac{1}{2}$

(d) $x_{n+1} = x_n^{\ 2}$ and $x_1 = 2$

(e) $x_{n+1} = 5x_n - 3$ and $x_1 = 0.25$

2. A school is expected to grow by 3% each year for the next five years.
The iteration $x_{n+1} = 1.03x_n$ is used to work out the number of pupils each year and $x_0 = 560$

(a) What does x_0 mean?

(b) Use the iteration to work out the number of pupils in five years' time.

3. It has been predicted that the population of a small village is to decline at a rate of 2.5% each year for the next 8 years.

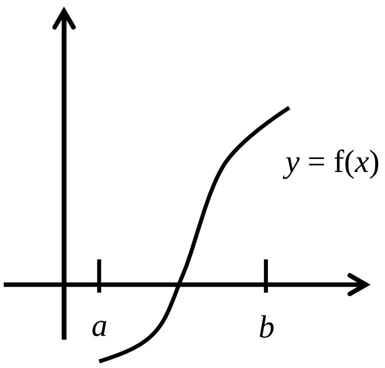
Use the iteration $x_{n+1} = 0.975x_n$ to find the population in four years time of a village whose present population is 1848.

LOCATING ROOTS

Suppose the graph of $y = f(x)$ is continuous and crosses the x-axis between $x = a$ and $x = b$.

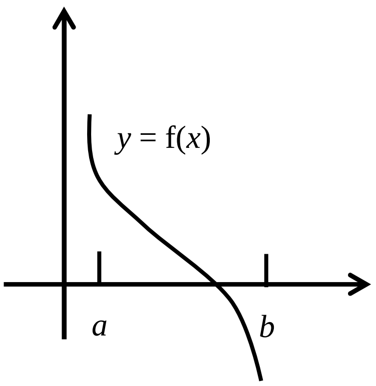
This situation can be seen in diagram 1 and in diagram 2.

Diagram 1



$f(a) < 0$ and $f(b) > 0$

Diagram 2



$f(a) > 0$ and $f(b) < 0$

In each case there is a **change in sign** for y either going from negative to positive (in Diagram 1) or from positive to negative positive (in Diagram 2).

If there is **a change in sign** for y for two particular values of x then we can say there is **a root** between these values of x and we can say that the equation $f(x) = 0$ will have a solution between these two values of x.

EXAMPLE 2

Show that the equation $3+4x-x^4=0$ has a solution between x = 1 and x = 2

As the equation is already equal to zero, we write it in function notation

$f(x)=3+4x-x^4$ ← Write f(x) = given expression

Given x = 1 and x = 2

$f(1)=3+4(1)-1^4=6>0$ ← Substitute for x = 1 in $f(x)=3+4x-x^4$

$f(2)=3+4(2)-2^4=-5<0$ ← Substitute for x = 2 in $f(x)=3+4x-x^4$

As f(1) > 0 i.e. positive and f(2) < 0 i.e. negative
there is a **change of sign**. ← You must write this statement

This shows that there is a solution between x = 1 and x = 2

EXAMPLE 3

Show that the equation $x^2-3x=-1$ has a root between x = 2 and x = 3

We need to make the equation equal to zero,

$x^2-3x+1=0$

Write the equation in function notation

$f(x)=x^2-3x+1$ ← Once f(x) = 0, write f(x) = given expression

Given x = 2 and x = 3

$f(2)=2^2-3(2)+1=-1<0$ ← Substitute for x = 2 in $f(x)=x^2-3x+1$

$f(3)=3^2-3(3)+1=1>0$ ← Substitute for x = 3 in $f(x)=x^2-3x+1$

As f(2) < 0 and f(3) > 0 there is a **change of sign**. ← You must write this statement

This shows that there is a root between x = 2 and x = 3

NOTE: We use the same method if asked for a **root** or a **solution** of the equation.

EXERCISE 2:

1. Show that the equation $x^3 = 14$ has a solution between $x = 2$ and $x = 3$

2. Show that the equation $x^3 + 6x^2 - 9x + 2 = 0$ has a solution between $x = 0$ and $x = 0.5$

3. Show that the equation $1 + x^2 - x^3 = 0$ has a solution between $x = 1$ and $x = 2$

4. Show that the equation $x^3 - x - 1 = 0$ has a solution between $x = 1$ and $x = 1.5$

5*. Does the equation $\frac{1}{x} = 0$ have a solution between $x = -1$ and $x = 0.5$?

REARRANGING EQUATIONS

In the exam you may be asked to rearrange an equation into an iteration formula. As before, basic algebraic rules apply.
Since these types of questions tend to be 'show that …', you must show all the intermediate steps to gain full marks.

EXAMPLE 4

Show that the equation $x^2 - 3x + 1 = 0$ can be arranged to give $x = \frac{x^2}{3} + \frac{1}{3}$

As we can see that there is x^2 and 1 in the 1st and 2nd equations we start by getting rid of the $-3x$ in the first equation. ← Look at what you are trying to show and see how it relates to what you are given

$x^2 - 3x + 1 = 0$ ← Start with the given equation

$x^2 + 1 = 3x$ ← Subtract $3x$ from each side

$\frac{x^2}{3} + \frac{1}{3} = x$ ← Divide each term by 3

Hence, $x = \frac{x^2}{3} + \frac{1}{3}$ ← Finish with a statement

EXAMPLE 5

Show that the equation $x^3 + 5x - 1 = 0$ can be arranged to give $x = \frac{1 - x^3}{5}$

We can see that 1 and x^3 are on the right and we need to divide by 5 ← Look at what you are trying to show and see how it relates to what you are given

$x^3 + 5x - 1 = 0$ ← Start with the given equation

$5x = 1 - x^3$ ← Isolate the term $5x$

$x = \frac{1 - x^3}{5}$ ← Divide each term by 5

Hence, $x = \frac{1 - x^3}{5}$ ← Finish with a statement

EXERCISE 3:

1. Show that the equation $x^2 + 4x - 6 = 0$ can be rearranged to give $x = \frac{3}{2} - \frac{x^2}{4}$

2. Show that the equation $x^2 - 5x + 6 = 0$ can be rearranged to give $x = \sqrt{5x - 6}$

3. Show that the equation $x^3 + x - 4 = 0$ can be rearranged to give $x = \sqrt[3]{4 - x}$

4. Show that the equation $x^3 + 3x^2 - 5 = 0$ can be rearranged to give $x = \sqrt[3]{5 - 3x^2}$

EXAMPLE 6

(a) Show that the equation $x^2 - 5x + 2 = 0$ has a root between $x = 4$ and $x = 5$

(b) Show that the equation $x^2 - 5x + 2 = 0$ can be arranged to give $x = \sqrt{5x - 2}$

(c) Use the iteration $x_{n+1} = \sqrt{5x_n - 2}$, with $x_0 = 5$, to find a solution to the equation $x^2 - 5x + 2 = 0$ correct to 1 decimal point.

(a) Write the equation in function notation

$$f(x) = x^2 - 5x + 2$$

As there is a root between $x = 4$ and $x = 5$

$f(4) = 4^2 - 5(4) + 2 = -2 < 0$ ← Substitute for $x = 4$ in $f(x) = x^2 - 5x + 2$

$f(5) = 5^2 - 5(5) + 2 = 2 > 0$ ← Substitute for $x = 5$ in $f(x) = x^2 - 5x + 2$

As f(4) < 0 and f(5) > 0
there is a **change of sign**.
so there is a root between $x = 4$ and $x = 5$ ← You must write this statement

(b) We can see that $5x$ and 2 are on the right and that we will need to square root ← Look at what you are trying to show and see how it relates to what you are given

$x^2 - 5x + 2 = 0$ ← Start with the given equation

$x^2 = 5x - 2$ ← add $5x$ and subtract 2 from each side

$x = \sqrt{5x - 2}$ ← square root both sides

(c) Given $x_{n+1} = \sqrt{5x_n - 2}$ and $x_0 = 5$

$x_1 = \sqrt{5x_0 - 2} = \sqrt{5(5) - 2}$
$= 4.7958 = 4.8$ to 1 d.p. ← 1st term $= \sqrt{5x_0 - 2}$ where $x_0 = 5$

$x_2 = \sqrt{5x_1 - 2} = \sqrt{5(4.7958) - 2}$
$= 4.6882 = 4.7$ to 1 d.p. ← 2nd term $= \sqrt{5x_1 - 2}$ where $x_1 = 4.7958$

$x_3 = \sqrt{5x_2 - 2} = \sqrt{5(4.6882) - 2}$
$= 4.6304 = 4.6$ to 1 d.p. ← 3rd term $= \sqrt{5x_2 - 2}$ where $x_2 = 4.6882$

$x_4 = \sqrt{5x_3 - 2} = \sqrt{5(4.6304) - 2}$
$= 4.5992 = 4.6$ to 1 d.p. ← 4th term $= \sqrt{5x_3 - 2}$ where $x_2 = 4.6304$

Hence, solution is **4.6** to 1 decimal place. ← We can stop as the rounding has settled to 4.6

NOTE: Always use your **unrounded** value when substituting.

EXAMPLE 7

(a) Show that the equation $3x^2 - x^3 + 3 = 0$ can be arranged to give $x = 3 + \frac{3}{x^2}$

(b) Using $x_{n+1} = 3 + \frac{3}{x_n^{\ 2}}$ with $x_0 = 3.2$, find the values of x_1, x_2 and x_3.

(c) Explain what the values of x_1, x_2 and x_3 represent.

(a) $3x^2 - x^3 + 3 = 0$ ← We can see we want $\frac{3}{x^2}$ so divide each term by x^2

$3 - x + \frac{3}{x^2} = 0$ ← Divide each term by x^2

$3 + \frac{3}{x^2} = x$ ← Add x to each side of the equation

Hence, $x = 3 + \frac{3}{x^2}$ ← You must write this statement

(b) Given $x_{n+1} = 3 + \frac{3}{x_n^{\ 2}}$ and $x_0 = 3.2$

$x_1 = 3 + \frac{3}{x_0^{\ 2}} = 3 + \frac{3}{3.2^2} = 3.293$ ← 1st term $= 3 + \frac{3}{x_0^{\ 2}}$ where $x_0 = 3.2$

$x_2 = 3 + \frac{3}{x_1^{\ 2}} = 3 + \frac{3}{3.293^2} = 3.277$ ← 2nd term $= 3 + \frac{3}{x_1^{\ 2}}$ where $x_1 = 3.277$

$x_3 = 3 + \frac{3}{x_2^{\ 2}} = 3 + \frac{3}{3.277^2} = 3.279$ ← 3rd term $= 3 + \frac{3}{x_2^{\ 2}}$ where $x_2 = 3.277$

(c) Each iteration is a better estimation of the solution of the equation $3x^2 - x^3 + 3 = 0$

NOTE: If you were asked to find a solution of the equation $3x^2 - x^3 + 3 = 0$ to 2 d.p. then the answer would be 3.28 as the value of x has settled to 3.28 to 2 d.p.

EXERCISE 3:

1. (a) Show that the equation $x^2 - x - 7 = 0$ can be rearranged to give $x = \sqrt{x+7}$

This information can be used to obtain the iterative formula $x = \sqrt{x+7}$

(b) Starting with $x_0 = 4$ calculate the values of x_1, x_2 and x_3, giving all the figures on your calculator display.

(c) Find one solution of $x^2 - x - 7 = 0$ correct to 3 decimal places.

2. (a) Show that the equation $x^2 + x - 3 = 0$ can be rearranged to give $x = \dfrac{3}{1+x}$

(b) Use the iteration $x_{n+1} = \dfrac{3}{1+x_n}$, with $x_0 = 1$, to find the values of x_1, x_2 and x_3.

3. (a) Show that the equation $x^3 + 4x = 1$ has a solution between $x = 0$ and $x = 1$

(b) Show that the equation $x^3 + 4x = 1$ can be arranged to give $x = \dfrac{1}{4} - \dfrac{x^3}{4}$

(c) Starting with $x_0 = 0$ use the iteration formula $x_{n+1} = \dfrac{1}{4} - \dfrac{x_n^{\;3}}{4}$ twice, to find an estimate for the solution of $x^3 + 4x = 1$

4. An approximate solution to an equation is found using this iterative process

$$x_{n+1} = \frac{x_n^{\;3} - 3}{8} \text{ and } x_0 = -1$$

Work out the values of x_1, x_2 and x_3.

5. Show that the equation $x^3 - 5x + 7 = 0$ can be rearranged to form the following iterative formulae.

(a) (i) $x = \dfrac{1}{5}\left(x^3 + 7\right)$ (ii) $x = \sqrt[3]{5x - 7}$

Use the following iterations where $x_0 = -2$ in each case to find the values of x_1, x_2 and x_3.

(b) (i) $x_{n+1} = \dfrac{1}{5}\left(x_n^{\;3} + 7\right)$ (ii) $x_{n+1} = \sqrt[3]{5x_n - 7}$

6. (a) Show that the equation $x^3 - 10x = 30$ has a solution between $x = 4$ and $x = 5$

(b) Show that the equation $x^3 - 10x = 30$ can be arranged to give $x = \sqrt[3]{30 + 10x}$

(c) Starting with $x_0 = 4.5$ use the iteration formula $x_{n+1} = \sqrt[3]{30 + 10x_n}$ to find an estimate for the solution of $x^3 - 10x = 30$ to 2 decimal places.

7. (a) Complete the table for $y = x^3 - 5x + 4$

x	0	1	2	3	4
y		−8			40

(b) Between which two consecutive integers is there a solution to the equation $x^3 - 5x + 4 = 0$? Give a reason for your answer.

(c) Show that the equation $x^3 - 5x + 4 = 0$ can be arranged to give $x = \sqrt[3]{5x+4}$

(d) Starting with $x_0 = 2.5$ use the iteration formula $x_{n+1} = \sqrt[3]{5x_n + 4}$ to find an estimate for the solution of $x^3 - 5x + 4 = 0$ to 2 decimal places.

8. (a) show that $x = 1 + \dfrac{11}{x-3}$ is a rearrangement of the equation $x^2 - 4x - 8 = 0$

(b) Use the iterative formula $x_{n+1} = 1 + \dfrac{11}{x_n - 3}$ together with a starting value of $x_0 = -2$ to obtain a solution of the equation $x^2 - 4x - 8 = 0$ correct to 1 decimal place.

9. A computer uses the iteration $x_{n+1} = \dfrac{2}{x_n + 3}$ to find one solution for a quadratic equation.

(a) What quadratic equation is being solved?

(b) Find the positive solution of this equation.

10. (a) Show that $x = \dfrac{5}{x-1}$ can be rearranged to give $x^2 - x - 5 = 0$

(b) Use the iterative formula $x_{n+1} = \dfrac{5}{x_n - 1}$ together with a starting value of $x_0 = -2$ to obtain a solution of the equation $x^2 - x - 5 = 0$

(c) The other solution is approximately 3.

What happens if we use 3 as the starting value in the iteration $x_{n+1} = \dfrac{5}{x_n - 1}$?

(d) Show that $x = \dfrac{x^2 + 5}{2x - 1}$ can be rearranged to give $x^2 - x - 5 = 0$

(e) Use the iterative formula $x_{n+1} = \dfrac{x_n^{\ 2} + 5}{2x_n - 1}$ together with a starting value of $x_0 = -2$ to obtain the other solution of the equation $x^2 - x - 5 = 0$

ANSWERS

Exercise 1

1. (a) 5, 7, 9, 11, 13, 15 (b) 5, 14, 41, 122, 365
(c) 0, −1, −3, −7, −15 (d) 4, 16, 256, 65536, 4294967296
(e) $-\frac{7}{4}, -\frac{47}{4}, -\frac{247}{4}, -\frac{1247}{4}, -\frac{6247}{4}$

2. (a) Number of pupils at the start (b) 649

3. 1713

Exercise 2

1. f(2) = −6 < 0 and f(3) = 13 > 0
Hence, change of sign; equation will have a solution between these two values of x

2. f(0) = 2 > 0 and f(0.5) = −0.875 < 0
Hence, change of sign; equation will have a solution between these two values of x

3. f(1) = 1 > 0 and f(2) = −3 < 0
Hence, change of sign; equation will have a solution between these two values of x

4. f(1) = −1 < 0 and f(1.5) = 0.875 > 0
Hence, change of sign; equation will have a solution between these two values of x

5*. f(−1) = −1 < 0 and f(0.5) = 2 > 0
Hence, change of sign; discuss with students why this equation will not have the a solution between these two values

Exercise 3

1. $x^2 + 4x - 6 = 0$
$4x = 6 - x^2$
$x = \frac{6}{4} - \frac{x^2}{4}$
$x = \frac{3}{2} - \frac{x^2}{4}$

2. $x^2 - 5x + 6 = 0$
$x^2 = 5x - 6$
$x = \sqrt{5x - 6}$

3. $x^3 + x - 4 = 0$
$x^3 = 4 - x$
$x = \sqrt[3]{4 - x}$

4. $x^3 + 3x^2 - 5 = 0$
$x^3 = 5 - 3x^2$

$$x = \sqrt[3]{5 - 3x^2}$$

Exercise 4

1. (a) $x^2 - x = 0$
$x^2 = x + 7$
$x = \sqrt{x+7}$

(b) $x_1 = 3.31662479$
$x_2 = 3.211950309$
$x_3 = 3.19561423$

(c) 3.193

2. (a) $x^2 + x - 3 = 0$
$x(x + 1) = 3$
$x = \dfrac{3}{1+x}$

(b) $x_1 = 1.5$
$x_2 = 1.2$
$x_3 = 1.364$

3. (a) $f(0) = -1 < 0$ and $f(1) = 4 > 0$
Hence, change of sign

(b) $x^3 + 4x = 1$
$x = 1 - x^3$
$x = \dfrac{1}{4} - \dfrac{x^3}{4}$

(c) 0.246

4. $x_1 = -0.5$
$x_2 = -0.391$
$x_3 = -0.382$

5. (a) (i) $x^3 - 5x + 7 = 0$
$5x = x^3 + 7$
$x = \frac{1}{5}(x^3 + 7)$

(ii) $x^3 - 5x + 7 = 0$
$x^3 = 5x - 7$
$x = \sqrt[3]{5x-7}$

(b)(i) $x_1 = -0.2$
$x_2 = 1.3984$
$x_3 = 1.947$

(ii) $x_1 = -2.571$
$x_2 = -2.708$
$x_3 = -2.739$

6. (a) $f(4) = -6 < 0$ and $f(5) = 45 > 0$

Hence, change of sign

(b) $x^3 - 10x = 30$

$x^3 = 30 + 10x$

$x = \sqrt[3]{30+10x}$

(c) 4.15

7. (a)

x	0	1	2	3	4
y	−4	−8	−6	8	40

(b) 2 and 3 due to the change of sign

(c) $x^3 - 5x - 4 = 0$

$x^3 = 5x + 4$

$x = \sqrt[3]{5x+4}$

(d) 2.56

8. (a) $x = 1 + \dfrac{11}{x-3}$

$x - 1 = \dfrac{11}{x-3}$

$(x - 1)(x - 3) = 11$

$x^2 - x - 3x + 3 = 11$

$x^2 - 4x - 8 = 0$

(b) −1.5

9. (a) $x^2 + 3x - 2 = 0$

(b) 0.56

10. (a) $x = \dfrac{5}{x-1}$

$x(x - 1) = 5$

$x^2 - x - 5 = 0$

(b) −1.79

(c) Obtains the first solution from part (b)

(d) $x = \dfrac{x^2+5}{2x-1}$

$x(2x - 1) = x^2 + 5$

$x^2 - x - 5 = 0$

(e) 2.79